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THE MATHEMATICS OF GROWTH AGAINST LIMITS

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ABSTRACT

The single step growth against limit equation is introduced and its consequences (s curves) are briefly explored. The possibility of sudden jumps (violations) from one growth limit curve to another is recognized and formulated. The full model featuring stochastic violations is developed. An integral equation governing the time evolution of probability density of growth is derived. The integral equation is reduced to a dispersion relation which is used to calculate the velocities of the first three moments of the growth probability density. The improvements in airliner performance over the years is analyzed by decomposition of certain economic data against the models parameters.

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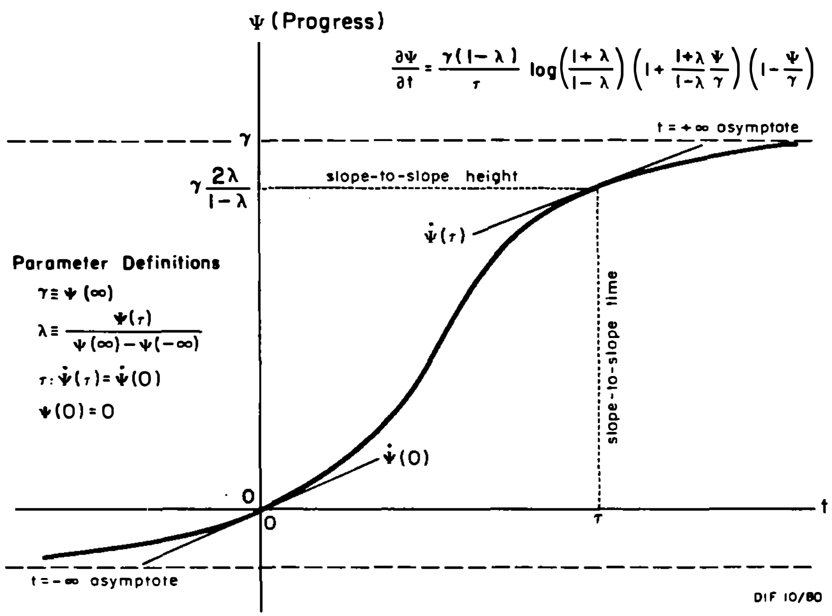
* I wish to thank Dr. Burton Klein for extensive discussion of the human underpinnings of all this.

THE MATHEMATICS OF GROWTH AGAINST LIMITS

Distinguish two varieties of cognitive discovery: "elaboration" - which is mapping and surveying within the span of ideas already extent, and "violation" - the unplanned intuitive leap that illuminates new realms. Elaboration yields progress which when charted opposite time resembles a skewed letter "s". This is the standard "growth against limits" curve known to bioligists. Driven stochastically by violations progress over longer times becomes an irregular staircase of "s" curves, each coupled continuously and smoothly with its predecessor.

Besides being circumscribed by the initial conditions of their birth points "s" curves (resulting from first order quadratic differential equations) require two additional parameters to be uniquely identified. Tau measures the time taken for the slope of an "s" curve to reattain the value it had at the birth point. Thus tau is the "slope-to-slope" time. Lambda governs the sharpness of an "s" curve. It is defined as the ratio of the slope-to-slope height to the "asymptote-to-asymptote" height of the curve.



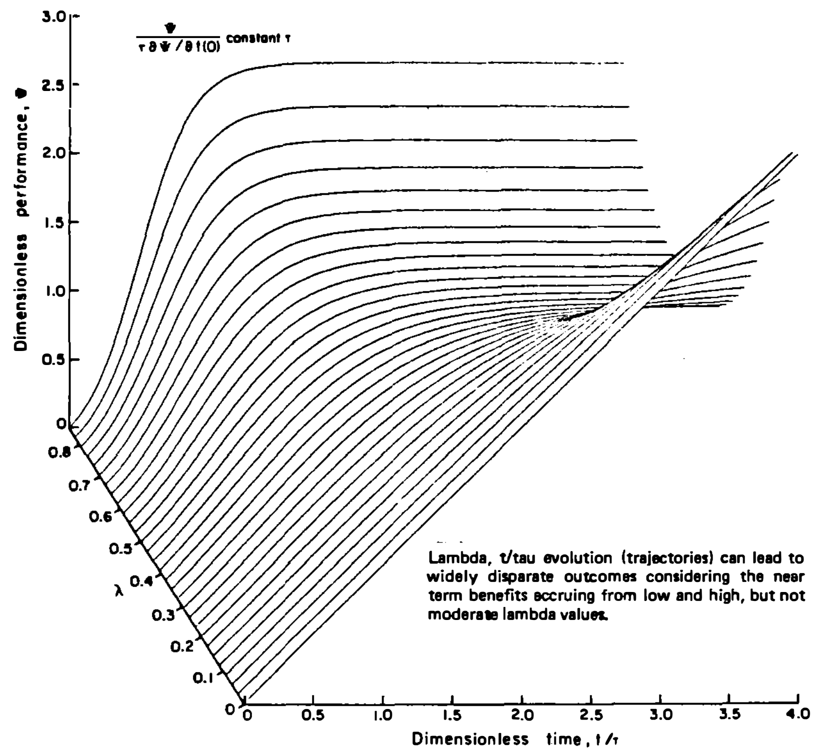
The constraint of smoothness at coupling points suggests we investigate the lambda,t properties of families of s curves all having the same initial slope. These are derived and plotted below.

$$\frac{d\,\Psi/\gamma}{d\,t/\tau} = (1-\lambda)\ln\left(\frac{1+\lambda}{1-\lambda}\right)\left(1+\frac{1+\lambda}{1-\lambda}\frac{\Psi}{\gamma}\right)\left(1-\frac{\Psi}{\gamma}\right)$$

$$\frac{1-\lambda}{2}\int_0^{\Psi}\left[\frac{\left(\frac{1+\lambda}{1-\lambda}\right)}{1+\left(\frac{1+\lambda}{1-\lambda}\right)\frac{\Psi}{\gamma}} + \frac{1}{1-\frac{\Psi}{\gamma}}\right]\frac{d\Psi}{\gamma} = (1-\lambda)\ln\left(\frac{1+\lambda}{1-\lambda}\right)\int_0^t\frac{dt}{\tau}$$

$$\ln\left(\frac{1+\left(\frac{1+\lambda}{1-\lambda}\right)\frac{\Psi}{\gamma}}{1-\frac{\Psi}{\gamma}}\right) = 2\ln\left(\frac{1+\lambda}{1-\lambda}\right)\frac{t}{\tau} \quad \rightarrow \quad \frac{1+\left(\frac{1+\lambda}{1-\lambda}\right)\frac{\Psi}{\gamma}}{1-\frac{\Psi}{\gamma}} = \left(\frac{1+\lambda}{1-\lambda}\right)^{2t/\tau}$$

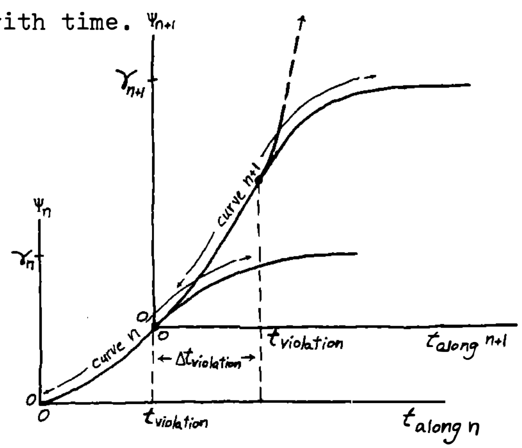
$$\frac{\Psi}{\gamma} = \frac{\left(\frac{1+\lambda}{1-\lambda}\right)^{2t/\tau} - 1}{\left(\frac{1+\lambda}{1-\lambda}\right) + \left(\frac{1+\lambda}{1-\lambda}\right)^{2t/\tau}} \quad \longrightarrow \quad \frac{\Psi}{\gamma} = \frac{1-\left(\frac{1-\lambda}{1+\lambda}\right)^{2t/\tau}}{1+\left(\frac{1-\lambda}{1+\lambda}\right)^{2t/\tau-1}}$$



Lambda, t/tau evolution (trajectories) can lead to widely disparate outcomes considering the near term benefits accruing from low and high, but not moderate lambda values.

S curves couple continuously and smoothly at violation points. Let such a violation mark the origin of a "comoving" coordinate system against which the new s curve will be described. Continuity is thus established by fiat;- the time and height zero of the new s curve are aligned at the violation point along the old s curve. Smoothness, which is first derivative continuity, will emerge as the crucial condition governing the relation between successive s curves.

Inspection of the s curve defining differential equation reveals that the slope at any point along an s curve is proportional to the starting slope;- the proportionality factor involves lambda and t/tau, but not the initial derivative. Thus the starting slope of a new s curve is proportional to the starting slope of the preceding s curve, the proportionality factor depending only on lambda, tau and the time interval between the violations marking the beginnings of the old and new curves. Lastly note that if lambda and tau are to be held constant than the parameter gamma of each new s curve must be adjusted to achieve equality of slopes at violations. Further the starting slope of a curve is proportional to that curve's gamma. Thus all slope proportionalities above translate directly into gamma proportionalities and the conclusion emerges that <u>gamma evolves geometrically from curve to curve, the precise ratio varying, but depending only on lambda, tau and the time intervals between successive violations</u>. This suggests that we consider log(gamma) since we expect it to move roughly linearly with time.



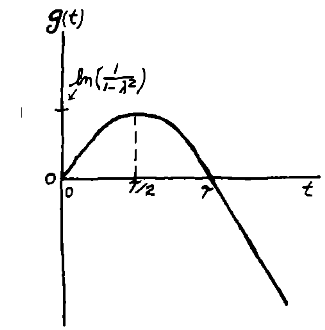
$$\dot{\Psi}_{n+1}(0) = \dot{\Psi}_n(t_{violation})$$
$$= \dot{\Psi}_n(0) \cdot f(\lambda, t_{violation}/\tau)$$

$$\gamma_{n+1} = \gamma_n \cdot f(\lambda, t_{violation}/\tau)$$

$$G_n \equiv \log \gamma_n \qquad g \equiv \log f$$

$$G_{n+1} = G_n + g(t_{violation})$$

Function g depends on the specifics of the s curve model. Our model results in a g symmetric in time about tau/2, having a single maximum at tau/2, crossing zero at tau (since $\dot{\Psi}(\tau)/\dot{\Psi}(0) = 1$) and becoming asymptotically linear for large t. Exactly:



$$g(t) = \log\left(\dot{\Psi}(t)/\dot{\Psi}(0)\right)$$

$$\dot{\Psi}(t)/\dot{\Psi}(0) = \left(1 + \frac{1+\lambda}{1-\lambda}\,\frac{1-\left(\frac{1-\lambda}{1+\lambda}\right)^{2t/\tau}}{1+\left(\frac{1-\lambda}{1+\lambda}\right)^{2t/\tau-1}}\right)\cdot\left(1-\frac{1-\left(\frac{1-\lambda}{1+\lambda}\right)^{2t/\tau}}{1+\left(\frac{1-\lambda}{1+\lambda}\right)^{2t/\tau-1}}\right)$$

$$= \left(\frac{1+\left(\frac{1-\lambda}{1+\lambda}\right)^{-1}}{1+\left(\frac{1-\lambda}{1+\lambda}\right)^{2t/\tau-1}}\right)\left(\frac{1+\left(\frac{1-\lambda}{1+\lambda}\right)^{-1}}{1+\left(\frac{1-\lambda}{1+\lambda}\right)^{2t/\tau-1}}\right)\left(\frac{1-\lambda}{1+\lambda}\right)^{2t/\tau}$$

$$= \left(\frac{\left(\frac{1-\lambda}{1+\lambda}\right)^{+\frac{1}{2}} + \left(\frac{1-\lambda}{1+\lambda}\right)^{-\frac{1}{2}}}{\left(\frac{1-\lambda}{1+\lambda}\right)^{\frac{t}{\tau}-\frac{1}{2}} + \left(\frac{1-\lambda}{1+\lambda}\right)^{\frac{1}{2}-\frac{t}{\tau}}}\right)^2 = \left((1-\lambda^2)\cosh^{+2}\left[\left(\frac{t}{\tau}-\frac{1}{2}\right)\ln\left(\frac{1-\lambda}{1+\lambda}\right)\right]\right)^{-1}$$

Thus violating consistantly before tau yields exponentially increasing gamma, while tardier performance results in exponentially fast decay of gamma to zero.

The vagaries of human creativity urge strongely the introduction of a stochastic element. "Genius like lightning strikes";- the precise timing of violations we leave in the hands of god along with the stipulation that the time intervals between violations obey a negative exponential distribution with mean rate α.

$P_n(T)dT \equiv$ probability that n[th] curve is born in (T,T+dT) Clearly if curve n is born near T then its immediate ancestor must have been born sometime prior to T. More exactly:

$$\underbrace{P_n(T)\,dT}_{P(n \text{ born near } T)} = \int_0^T \underbrace{P_{n-1}(t)\,dt}_{P(n-1 \text{ born near } T)\cdot} \underbrace{e^{-\alpha(T-t)}\,\alpha\,dT}_{P(\text{no more violations till } T)} \rightarrow \frac{dP_n(T)}{dT} + \alpha P_n(T) = \alpha P_{n-1}(T)$$

$$\text{and} \quad P_0(T) = \delta(T)$$

This is easily solved; it yields a poisson type distribution:

$$P_n(T) = \alpha\,\frac{(\alpha T)^{n-1}}{(n-1)!}\,e^{-\alpha T}$$

The growth of G (=log gamma) is seen to depend crucially on the violation timing which itself is a random variable. Thus G_n becomes a probability distribution. Notice the double uncertainty: both the birth time and the gamma value of the n^{th} curve have become distributions. Shortly we will slur the distinction on n (in effect by averaging over all possible n values) and conceive of G as a distribution evolving smoothly with time. What follows is a deduction of the time evolution equation of G's distribution. Essentially it derives of the mechanistic "single step" staircase construction outlined above.

$P_n(T) \equiv$ prob density for n^{th} curve to begin at T

$P_n(G|T) \equiv$ prob density for G_n <u>given</u> that n^{th} curve began at T

Turning the mechanistic description around we see that for a certain G_n to happen at T we require a special G_{n-1}, namely one which when augmented by g(latest violation time interval) equals G_n. Thus $G_{n-1}(t)+g(\Delta t_{violation})=G_n(t+\Delta t_{violation})$.

$$\underbrace{P_n(T)dT\, P_n(G|T)dG}_{\substack{\text{P(n happens near T}\\ \text{with } G_n \text{ near G)}}} = \int_0^T \underbrace{P_{n-1}(t)dt\, P_{n-1}(G-g(T-t)|T)dG}_{\substack{\text{P(n-1 happens near t, before T,}\\ \text{with } G_{n-1} \text{ near G-g)}}}\; \underbrace{e^{-\alpha(T-t)}\alpha dT}_{\substack{\text{P(n-1 spawns n}\\ \text{near T)}}}$$

Since we know $P_n(t)$ in principle we have $P_n(G|T)$. We may simplify things considerably by slurring n:

$\mathcal{P}(G|T) \equiv$ prob density for G given that newest violation was at T

$$\mathcal{P}(G/T) = \sum_{n=1}^{\infty} P_n(T)\, P_n(G/T)$$

So we sum the integral equation on n from 2 to ∞. Also we change variables t'=T-t in the integral then erase the primes.

$$\mathcal{P}(G/T) - P_1(G/T)\,\alpha e^{-\alpha T} = \int_0^T \mathcal{P}(G-g(t)/T-t)\, e^{-\alpha t}\alpha dt$$

Consider the G,T plane. This equation says that in order to get to (G,T) one may either hop there from some other point (G-g(t),T-t) in one step, or one must jump there from outside

the domain in one step. As T gets large $P_1(T)\to 0$; the possibility of jumping to (G,T) from outside becomes small. Specialize to this, long time "pure" evolution equation. Also substitute nu/tau for alpha; nu is a dimensionless frequency, it is the average number of violations per tau.

$$\mathcal{P}(G|T) = \int_0^\infty \mathcal{P}(G-g(t)|T-t)\, e^{-\nu t/\tau}\, \nu dt/\tau$$

Since this is a linear relation we can fourier transform it on G and on T. Looking ahead, what will emerge will be a functional dependence of the transformed variables upon one another;- a dispersion relation. Thus we intuit:

$$\mathcal{P}(G/T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\mathcal{P}}(k)\, e^{i(kG+\omega T/\tau)}\, dk$$

plugging this into our integral equation:

$$\int_{-\infty}^{\infty} \hat{p}(k)\, e^{i(kG+\omega T/\tau)} dk = \int_{-\infty}^{\infty} \hat{p}(k)\, e^{i(kG+\omega T/\tau)} dk \left\{ \int_0^\infty e^{-i(kg(t)+\omega t/\tau)}\, e^{-\nu t/\tau}\, \nu dt/\tau \right\}$$

changing variables: $x=\nu t/\tau$ we conclude that:

$$1 = \int_0^\infty e^{-i(kg(x\tau/\nu)+\omega x/\nu)-x}\, dx \qquad \text{"dispersion relation"}$$

$$\omega \longrightarrow \omega(k)$$

also $k=0 \to \omega=0$

We descend from these mathematical heights to somewhat nearer reality by computing from our solution the time evolution of the mean, varience and (for data analysis) something related to the third moment of the conditional probability distribution $\mathcal{P}(G|T)$. The computation uses the definitions of these quantities and some delta function manipulation.

$$\int_{-\infty}^{\infty} p(G|T)\,dG = 1 \longrightarrow 1 = \int_{-\infty}^{\infty} dk\ \hat{p}(k)\, e^{i\omega(k)T/\tau}\, \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikG}\, dG$$

$$= \int_{-\infty}^{\infty} dk\ \hat{p}(k)\,\delta(k) = \boxed{\hat{p}(0) = 1}$$

More generally;

$$\langle G^n\rangle(T) = \int_{-\infty}^{\infty} G^n\, p(G|T)\, dG = \int_{-\infty}^{\infty} dk\ \hat{p}(k)\, e^{i\omega(k)T/\tau}\, \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} G^n e^{ikG}\, dG}_{\frac{d^n}{d(ik)^n}\delta(k)}$$

integrating by parts n times on k;

$$\langle G^n\rangle(T) = \frac{(-1)^n}{i^n}\frac{d^n}{dk^n}\left\{\hat{p}(k)\, e^{i\omega(k)T/\tau}\right\}\Bigg]_{k=0}$$

$$\langle G\rangle(T) = \frac{-1}{i}\frac{d}{dk}\ \hat{p}(k)\, e^{i\omega(k)T/\tau}\Big|_{k=0} = i\hat{p}'(0) - \frac{T}{\tau}\frac{d\omega}{dk}(0) = \langle G\rangle(0) - \frac{T}{\tau}\frac{d\omega}{dk}(0)$$

$$\boxed{\frac{d\langle G\rangle}{dT/\tau} = -\frac{d\omega}{dk}(0)}$$ *"group velocity" of the distribution*

In a bit we'll evaluate $d\omega/dk(0)$. Evidently $\langle G\rangle$ moves linearly with time, so "gamma" changes exponentially, as advertised. In a similar manner we evaluate $\langle G^2\rangle$ and $\langle G^3\rangle$, the results will be presented shortly.

By repeatedly differentiating the dispersion relation we relate the various derivatives of ω upon k at k=0 to integrals of the shape factor g(t).

$$0 = \int_0^{\infty}\left(g\left(\frac{x\tau}{\nu}\right) + \frac{x}{\nu}\frac{d\omega}{dk}\right) e^{-i\left(kg\left(\frac{x\tau}{\nu}\right) + \frac{x}{\nu}\omega\right) - x}\, dx$$

$$\boxed{\frac{d\omega}{dk}(0) = -\nu\int_0^{\infty} g\left(\frac{x\tau}{\nu}\right) e^{-x}\, dx}$$

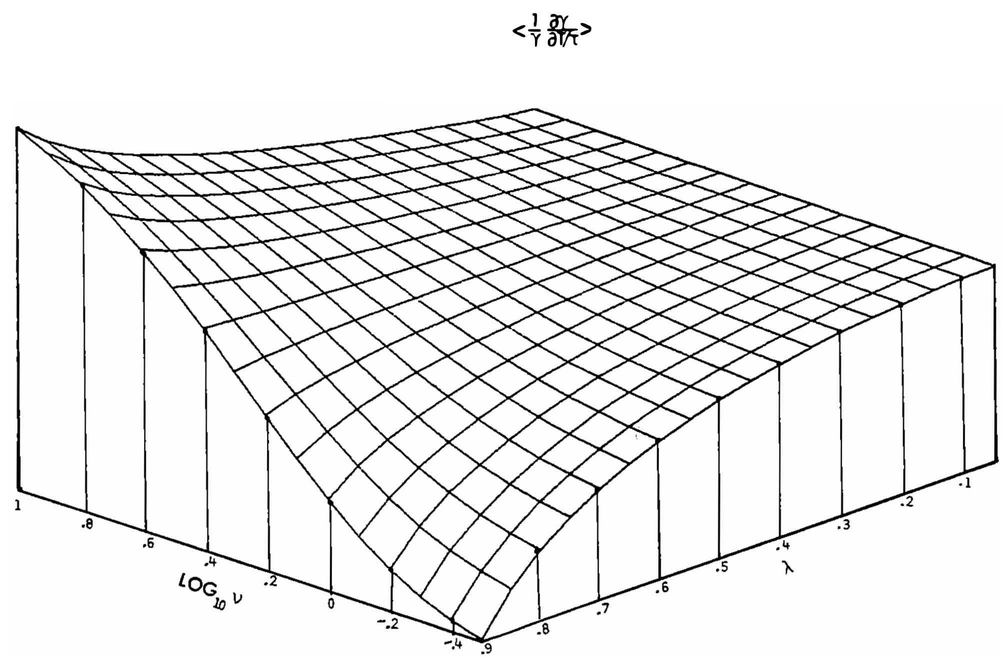
Combining all this reveals the essence of the statistical model: the temporal rates of change of the moments of the log(gamma)=G distribution.

$$\frac{d\langle G\rangle}{dT/\tau} = -\frac{d\omega}{dk} = \nu\int_0^{\infty} g\left(\frac{x\tau}{\nu}\right) e^{-x}\, dx$$

$$\frac{d\langle (G-\langle G\rangle)^2\rangle}{dT/\tau} = \frac{1}{i}\frac{d^2\omega}{dk^2} = \nu\int_0^{\infty}\left[g\left(\frac{x\tau}{\nu}\right) + \frac{x}{\nu}\frac{d\omega}{dk}(0)\right]^2 e^{-x}\, dx$$

$$\frac{d\langle (G-\langle G\rangle)^3 + (\tau\frac{d\langle G\rangle}{dT})^3\rangle}{dT/\tau} = \frac{d^3\omega}{dk^3} = \nu\int_0^{\infty}\left[-\left(g+\frac{x}{\nu}\omega'\right)^3 + \frac{3x}{\nu}\frac{\omega''}{i}\left(g+\frac{x}{\nu}\omega'\right)\right] e^{-x}\, dx$$

This completes the formal developement of the growth against limits model.

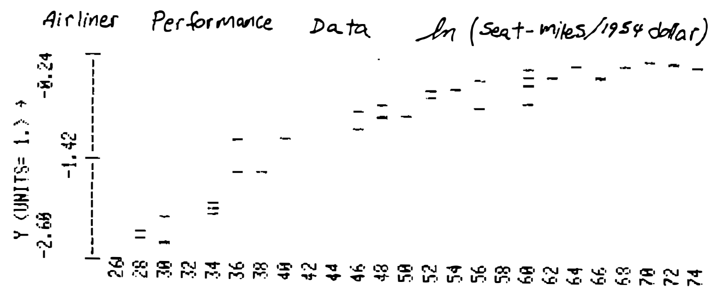


Data Analysis
(data on next page)

For each period the first half of the data is used to establish a trend against which the last half is differenced. The differences when squared (or whatever) are then least squares fitted. Thus the time derivatives of the moments are deduced.

The "M" numbers are essentially these time derivatives. Exactly, each "M" is the slope of a least square line through the appropriately processed data. Moments ("M"'s actually) one and two relate directly to the mean and varience. Moment 3 is the linear part of the true third moment.

The ratios of successive moments ("M"'s) are time scale invarient and so are functions of the two dimensionless parameters lambda and nu. Finding the lambda and nu associated with a given ratio set M2/M1, M3/M2 is practically messy (needs a computer), but straightforward in principle. It amounts to solving a pair of nonlinear transcendental equations in two unknowns. The results displayed were found by interpolation off a moderately course (20 X 10 entries) table covering an appropriate range of lambda, nu.



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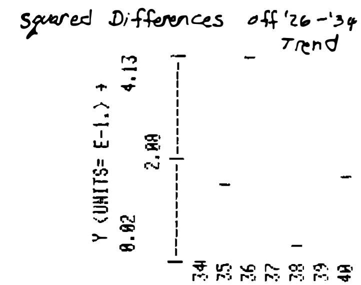

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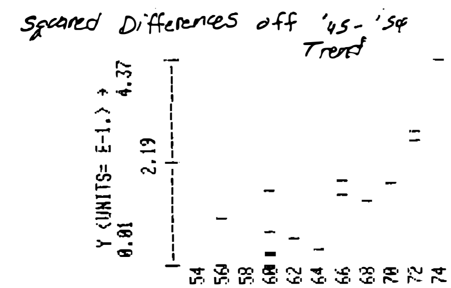

Early Years ($\lambda = .51$, $\nu = 4.5$, $\tau = 7.5$)   Later Years ($\lambda = .49$, $\nu = 3.0$, $\tau = 17$)